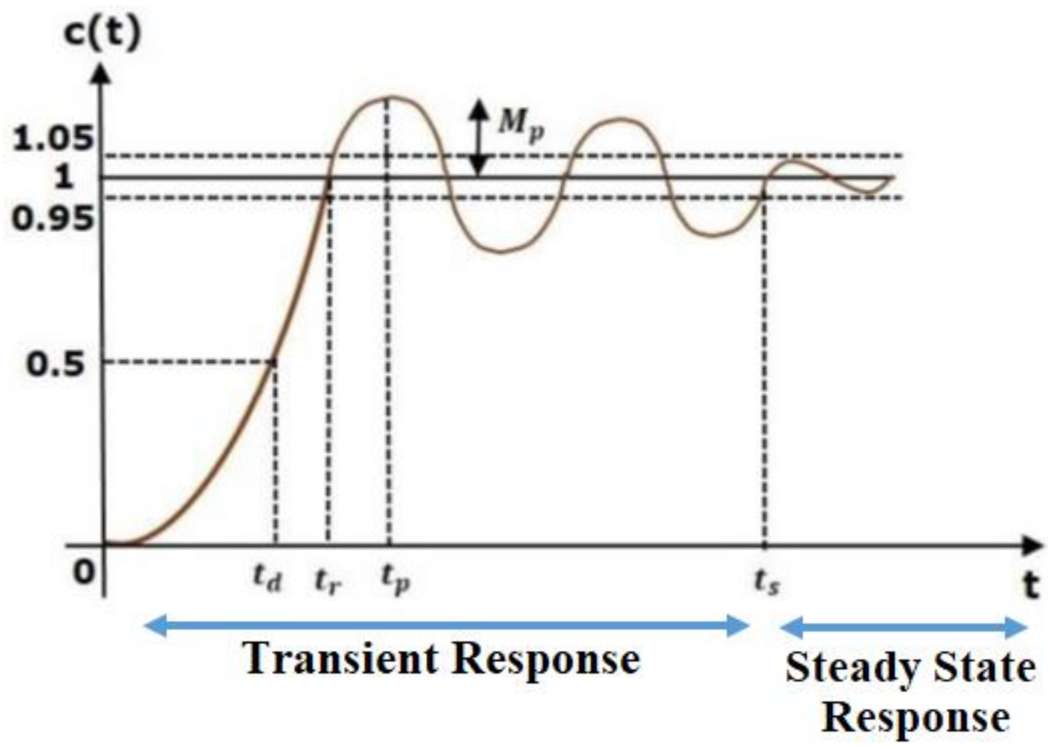
6.4 Time Domain Specifications for A Second-Order Under Damped Systems:

The transient response of a practical control system often exhibits damped oscillations before reaching steady state. The step response of the second-order system for the under damped case is shown in the following figure.



The response up to the settling time (t_s) is known as transient response and the response after the settling time is known as steady state response. All the time domain specifications which represented in this figure are:

1. **Delay Time (t_d)**: This is the time required for c(t) to reach half of its final value.

   Consider the step response of the second-order system for $t \geq 0$, when $0 < \zeta < 1$.

$$c(t) = \left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)\right) u(t)$$

First, we must find the final value of the step response as follows:

$$c(\infty) = \left(1 - \frac{e^{-\zeta\omega_n(\infty)}}{\sqrt{1-\zeta^2}}\sin(\omega_d(\infty) + \theta)\right) u(\infty)$$

But $u(\infty) = 1$ and $e^{-\zeta\omega_n(\infty)} = e^{-\infty} = \frac{1}{\infty} = 0$

$$c(\infty) = (1 - 0)\,(1) = 1$$

So, the final value of the step response is one.

Therefore, at t=t_d, the value of the step response will be 0.5. Substitute, this value in the above equation and replace t by t_d, we get:

$$c(t_d) = 0.5 = 1 - \frac{e^{-\zeta\omega_n t_d}}{\sqrt{1-\zeta^2}} sin(\omega_d t_d + \theta)$$

$$\left(\frac{e^{-\zeta\omega_n t_d}}{\sqrt{1-\zeta^2}}\right)\sin(\omega_d t_d + \theta) = 0.5$$

By using linear approximation, you will get the **delay time t_d** as:

$$\boldsymbol{t_d = \frac{1 + 0.7\zeta}{\omega_n}}$$

2. **Rise Time (t_r)**: It is the time required for the response to rise from 10% to 90%, 5% to 95%, or 0% to 100% of its final value. For under damped second order systems, the 0% to 100% rise time is normally used. For overdamped systems, the 10% to 90% rise time is commonly used.

We know the step response of the second-order system for t ≥ 0, when $0 < \zeta < 1$ is as follows:

$$c(t) = \left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)\right)\,u(t)$$

First, we must find the final value of the step response as follows:

$$c(\infty) = \left(1 - \frac{e^{-\zeta\omega_n(\infty)}}{\sqrt{1-\zeta^2}}\sin(\omega_d(\infty) + \theta)\right)\,u(\infty)$$

But $u(\infty) = 1$ and $e^{-\zeta\omega_n(\infty)} = e^{-\infty} = \frac{1}{\infty} = 0$

$$c(\infty) = (1 - 0)\,(1) = 1$$

So, the final value of the step response is one.

Therefore, at t=t_2, the value of step response is one. Substitute, these values in the following equation.

$$c(t) = \left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)\right) u(t)$$

$$c(t) = 1 = 1 - \frac{e^{-\zeta\omega_n t_2}}{\sqrt{1-\zeta^2}} sin(\omega_d t_2 + \theta)$$

$$\frac{e^{-\zeta\omega_n t_2}}{\sqrt{1-\zeta^2}} sin(\omega_d t_2 + \theta) = 0$$

$$sin(\omega_d t_2 + \theta) = 0$$

$$(\omega_d t_2 + \theta) = \pi$$

$$t_2 = \frac{\pi - \theta}{\omega_d}$$

Substitute t_1 and t_2 values in the following equation of rise time.

$$t_r = t_2 - t_1$$

where $t = t_1 = 0$

$$\boldsymbol{t_r = \frac{\pi - \theta}{\omega_d}}$$

From above equation, we can conclude that the rise time t_r and the damped frequency ω_d are inversely proportional to each other.

3. **Peak Time t_p:** It is the time required for the response to reach the **peak value** for the first time. It is denoted by $\mathbf{t_p}$ and can be obtained by differentiating c(t) with respect to time and letting this derivative equal zero. We know the step response of second order system for under-damped case is:

$$c(t) = \left(1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\sin(\omega_d t + \theta)\right) u(t)$$

Differentiate c(t) with respect to t, yields:

$$\frac{dc(t)}{dt} = \left(\left(-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\left(\omega_d \cos(\omega_d t + \theta)\right)\right) + \left((\sin(\omega_d t + \theta))\left(-\frac{-\zeta\omega_n e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\right)\right)$$

Substitute, $t=t_p$ and $\frac{dc(t)}{dt} = 0$ in the above equation, yields:

$$0 = \left(\left(-\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\right)\left(\omega_d \cos\left(\omega_d t_p + \theta\right)\right)\right) + \left(\left(\sin\left(\omega_d t_p + \theta\right)\right)\left(-\frac{-\zeta\omega_n e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\right)\right)$$

$$0 = -\frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\omega_d \cos\left(\omega_d t_p + \theta\right) + \frac{\zeta\omega_n e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}}\sin\left(\omega_d t_p + \theta\right)$$

Multiply both sides by $-\frac{\sqrt{1-\zeta^2}}{e^{-\zeta\omega_n t_p}}$, yields:

$$0 = \omega_d \cos\left(\omega_d t_p + \theta\right) - \zeta\omega_n \sin\left(\omega_d t_p + \theta\right)$$

But $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$$\omega_n\sqrt{1-\zeta^2}\, cos\left(\omega_d t_p + \theta\right) - \zeta\omega_n\, sin\left(\omega_d t_p + \theta\right) = 0$$

Divide both sides by ω_n, yields:

$$\sqrt{1-\zeta^2}\, cos\left(\omega_d t_p + \theta\right) - \zeta\, sin\left(\omega_d t_p + \theta\right) = 0$$

But $\sin(\theta) = \sqrt{1-\zeta^2}$ and $\zeta = \cos(\theta)$

$$\sin(\theta)\, cos\left(\omega_d t_p + \theta\right) - \cos(\theta)\ sin\left(\omega_d t_p + \theta\right) = 0$$

We know:

$$\sin(x - y) = sin(x)\cos(y) - \cos(x)\sin(y)$$

$$\sin\left(\theta - (\omega_d t_p + \theta)\right) = 0$$

$$\sin\left(-\omega_d t_p\right) = 0$$

$$\sin(x) = 0\ if\ x = 0, \pm\pi$$

$$-\omega_d t_p = -\pi$$

$$\boldsymbol{t_p = \frac{\pi}{\omega_d}}$$

From the above equation, we can conclude that the peak time t_p and the damped frequency ω_d are inversely proportional to each other.

4. **Peak Overshoot (M_p):** Peak overshoot is defined as the deviation of the response at peak time from the final value of response. It is also called the **maximum overshoot**.

   Mathematically, we can write it as: $M_p = c(t_p) - c(\infty)$

Where, $c(t_p)$ is the peak value of the response, $c(\infty)$ is the final (steady state) value of the response.

At $t=t_p$, the response c(t) is:

$$c(t_p) = 1 - \frac{e^{-\zeta\omega_n t_p}}{\sqrt{1-\zeta^2}} sin(\omega_d t_p + \theta)$$

Substitute, $t_p = \frac{\pi}{\omega_d}$ in the right-hand side of the above equation.

$$c(t_p) = 1 - \frac{e^{-\zeta\omega_n\left(\frac{\pi}{\omega_d}\right)}}{\sqrt{1-\zeta^2}} sin\left(\omega_d\left(\frac{\pi}{\omega_d}\right) + \theta\right)$$

We know that: $\sin(x+y) = sin(x)\cos(y) + \cos(x)\sin(y)$. So, we will get:

$$\sin(\pi+\theta) = sin(\pi)\cos(\theta) + \cos(\pi)\sin(\theta) = (0)\cos(\theta) + (-1)\sin(\theta) = -\sin(\theta)$$

$$\therefore c(t_p) = 1 - \frac{e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}}{\sqrt{1-\zeta^2}}(-sin(\theta))$$

We know that: $sin(\theta) = \sqrt{1-\zeta^2}$. So, we will get $c(t_p)$ as:

$$c(t_p) = 1 + e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}$$

Substitute the value of $c(t_p)$ and $c(\infty)$ in the peak overshoot equation.

$$M_p = c(t_p) - c(\infty) = \left(1 + e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}\right) - 1$$

$$\boldsymbol{M_p = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}}$$

Since the final steady-state value of the step response differs from unity, then it is common to use the maximum percent overshoot. It can be calculated by using this formula:

$$\boldsymbol{Maximum\ Percent\ Overshoot} = \frac{\boldsymbol{M_p}}{\boldsymbol{c}(\infty)} \times \mathbf{100\%}$$

The amount of the maximum (percent) overshoot directly indicates the relative stability of the system. We can conclude that the maximum percent overshoot will decrease if the damping ratio ζ increases.

5. **Settling Time (t_s):** It is the time required for c(t) to reach and remain within a certain range of its final value. This range is usually from 2–5% of the amplitude of the final value.

The settling time for 5% tolerance band (5% criterion) is:

$$\boldsymbol{t_s = \frac{3}{\zeta\ \omega_n} = 3\tau}$$

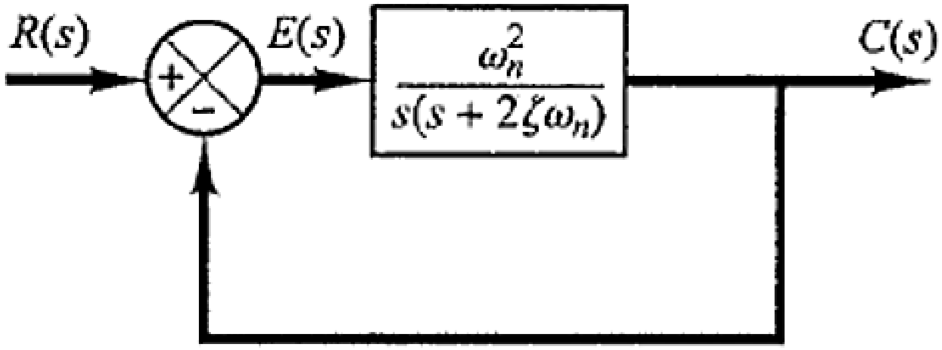
The settling time for 2% tolerance band is:

$$\boldsymbol{t_s = \frac{4}{\zeta\ \omega_n} = 4\tau}$$

Where, τ is the time constant and is equal to $\frac{1}{\zeta\ \omega_n}$.

- Both the settling time t_s and the time constant τ are inversely proportional to the damping ratio ζ.
- Both the settling time t_s and the time constant τ are independent of the system gain. That means even the system gain changes, the settling time t_s and time constant τ will never change.

Example1: Consider the system shown in Figure below, where ζ= 0.6 and $\boldsymbol{\omega_n}$ = 5 rad/sec. Find all the time domain specifications when the system is subjected to a unit-step input.



Solution:

The delay time t_d is:

$$t_d = \frac{1+0.7\zeta}{\omega_n} = \frac{1+(0.7\times 0.6)}{5} \Longrightarrow \boldsymbol{t_d = 0.284}$$

The rise time t_r is:

$$t_r = \frac{\pi-\theta}{\omega_d}$$

For the given values of ζ and ω_n, we obtain:

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 5\times\sqrt{1-0.6^2} \Longrightarrow \boldsymbol{\omega_d = 4\ rad/sec}.$$

$$And\ \because \sqrt{1-\zeta^2} = \sin(\theta) \Longrightarrow \theta = sin^{-1}\sqrt{1-\zeta^2} = sin^{-1}\sqrt{1-0.6^2}$$

$$\boldsymbol{\therefore \theta = 0.93\ rad}$$

$$t_r = \frac{\pi-\theta}{\omega_d} = \frac{3.14-0.93}{4} \Longrightarrow \boldsymbol{t_r = 0.55\ sec}$$

The peak time t_p is:

$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} \Longrightarrow \boldsymbol{t_p = 0.785\ sec}$$

The maximum percent overshoot is:

$$M_p = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} = e^{-\left(\frac{0.6\times 3.14}{\sqrt{1-0.6^2}}\right)} \Longrightarrow \boldsymbol{M_p = 0.095}$$

The maximum percent overshoot is thus

$$\boldsymbol{Maximum\ Percent\ Overshoot} = \frac{M_p}{c(\infty)}\times 100\% = \frac{0.095}{1}\times 100\% = \boldsymbol{9.5\%}$$

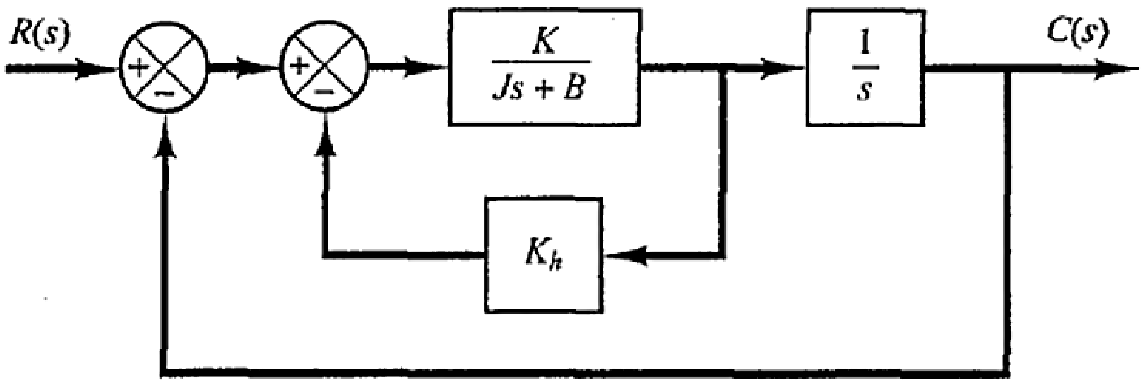
The settling time t_s (for 2% criterion) is:

$$t_s = \frac{4}{\zeta\, \omega_n} = \frac{4}{0.6 \times 5} \Rightarrow \boldsymbol{t_s = 1.333\ sec}$$

The settling time t_s (for 5% criterion) is:

$$t_s = \frac{3}{\zeta\, \omega_n} = \frac{3}{0.6 \times 5} = \frac{3}{3} \Rightarrow \boldsymbol{t_s = 1\ sec}$$

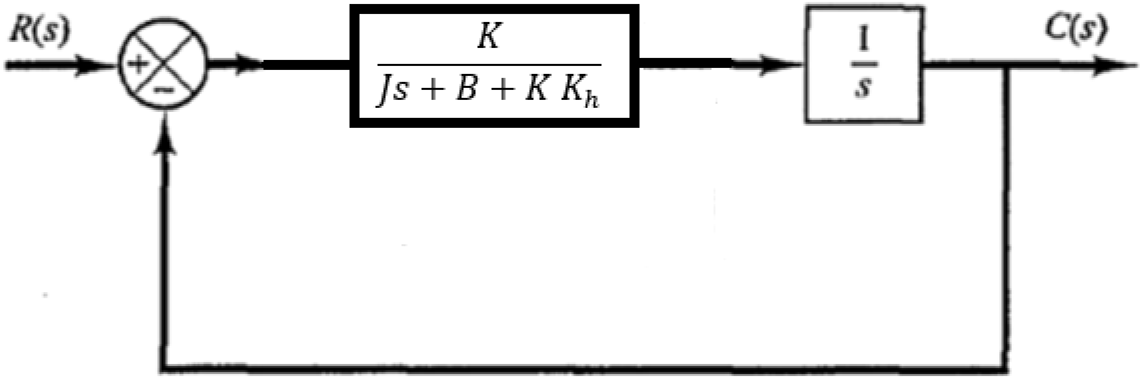
Example 2: **For the system shown in figure below, determine the values of gain K and velocity feedback constant K_h so that the maximum overshoot in the unit-step response is 0.2 and the peak time is 1 sec. With these values of K and Kh obtain the rise time and settling time. Assume that J = 1 kg-m^2 and B = 1 N-m/rad/sec.**



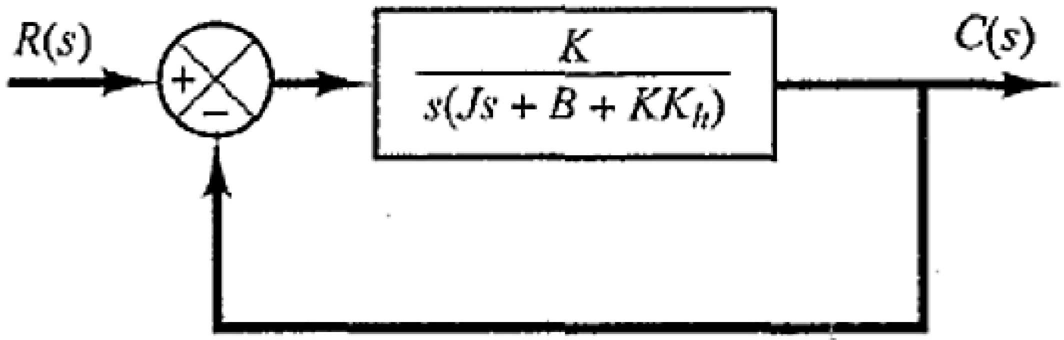
Solution:

For simplifying the block diagram in above figure, use **Rule 1** and **Rule 3** of block diagram reduction rules, as follows:

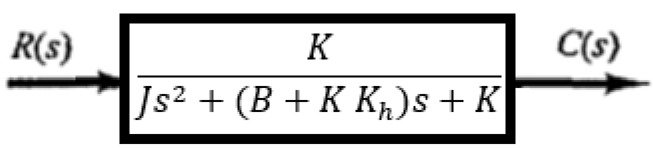
Step1: Use **Rule 3** for blocks $\frac{K}{Js+B}$ and K_h which are connected in negative feedback loop to combine them into one block. The modified block diagram is shown in the following figure.



Step2: Use **Rule 1** for blocks $\frac{K}{Js+B+KK_h}$ and $\frac{1}{s}$ which are connected in series to combine them into one block. The modified block diagram is shown in the following figure.



Step3: Use **Rule 3** for blocks $\frac{K}{Js+B+KK_h}$ and unity feedback which are connected in negative feedback loop to combine them into one block. After simplification, the block diagram of the system will be shown in figure below:



$$\frac{C(s)}{R(s)} = \frac{K}{Js^2 + (B + K\,K_h)s + K}$$

Or

$$\frac{C(s)}{R(s)} = \frac{K/J}{s^2 + \frac{(B + K\,K_h)}{J}s + K/J}$$

Comparing the above equation with the standard second order transfer function shown below:

$$\frac{C(s)}{R(s)} = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

The undamped natural frequency becomes: ${\omega_n}^2 = \frac{K}{J} \Longrightarrow \boldsymbol{\omega_n} = \sqrt{\boldsymbol{\frac{K}{J}}}$

The damping ratio ζ becomes: $\zeta = \frac{B+K\,K_h}{2J\sqrt{\frac{K}{J}}} \Longrightarrow \boldsymbol{\zeta} = \boldsymbol{\frac{B+K\,K_h}{2\sqrt{K\,J}}}$

The maximum overshoot M_p is:

$$M_p = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} = 0.2$$

Take ln for both sides, we get:

$$ln\left(e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} = 0.2\right)$$

$$= -\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = -1.609 \Longrightarrow \ \zeta\pi = 1.609\sqrt{1-\zeta^2}$$

Square both sides of the equation above, we get:

$$(\zeta\pi)^2 = (1.609\sqrt{1-\zeta^2})^2 \Longrightarrow (\zeta\pi)^2 + (1.609)^2\,\zeta^2 = (1.609)^2$$

$$12.448\,\zeta^2 = (1.609)^2 \Longrightarrow \zeta^2 = \frac{(1.609)^2}{12.448}$$

$\boldsymbol{\therefore \zeta = 0.456}$

$\because t_p = \frac{\pi}{\omega_d} = 1 \Longrightarrow \boldsymbol{\therefore \omega_d = 3.14\,rad/sec}$

$\because \omega_d = \omega_n\sqrt{\zeta^2 - 1} \Longrightarrow \omega_n = \frac{\omega_d}{\sqrt{\zeta^2-1}} = \frac{3.14}{\sqrt{0.456^2 - 1}} \Longrightarrow \boldsymbol{\omega_n = 3.53\,rad/sec}$

Since the natural frequency $\omega_n = \sqrt{\frac{K}{J}}$

Square both sides of the equation above, we get:

$$K = J{\omega_n}^2 = 1\,kg - m^2 \times (3.53rad/sec)^2$$

$\therefore K = 12.5\ N - m$

Since the damping ratio $\zeta = \frac{B+K\,K_h}{2\sqrt{K\,J}} \Longrightarrow 2\sqrt{K\,J}\,\zeta = B + K\,K_h \Longrightarrow 2\sqrt{K\,J}\,\zeta - B = K\,K_h$

$$\therefore K_h = \frac{2\sqrt{K\,J}\,\zeta - B}{K} = \frac{2\sqrt{12.5 \times 1}\ \times 0.456 - 1}{12.5}$$

$$\therefore \boldsymbol{K_h = 0.178\ sec}$$

$$\because \theta = sin^{-1}\sqrt{1-\zeta^2} = sin^{-1}\sqrt{1-0.456^2} = 1.10\ rad$$

$$t_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - 1.10}{4} \Longrightarrow \boldsymbol{t_r = 0.65\ sec}$$

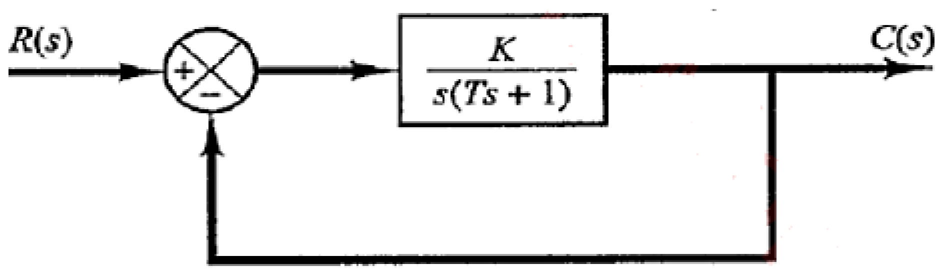
For the *2%* criterion, the settling time t_s is:

$$t_s = \frac{4}{\zeta\,\omega_n} = \frac{4}{0.456 \times 3.53} \Longrightarrow \boldsymbol{t_s = 2.48\ sec}$$

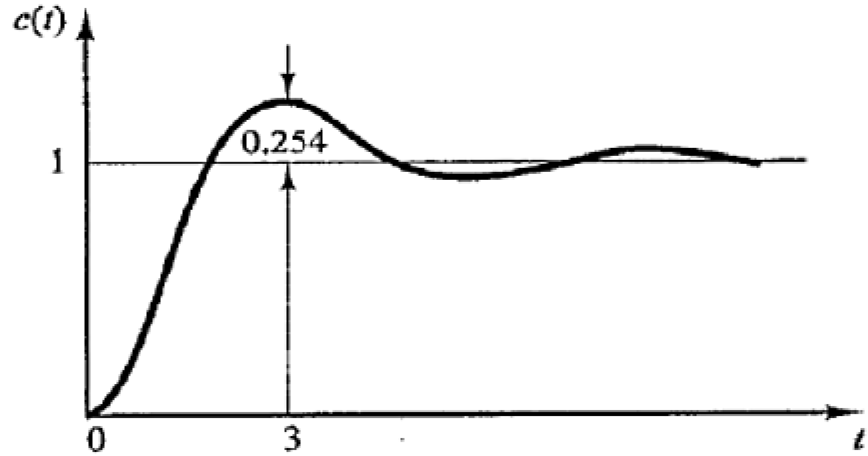
For the 5% criterion, the settling time t_s is:

$$t_s = \frac{3}{\zeta\,\omega_n} = \frac{3}{0.456 \times 3.53} \Longrightarrow \boldsymbol{t_s = 1.86\ sec}$$

Example 3: When the system shown in figure (a) is subjected to a unit-step input, the system output responds as shown in figure (b). Determine the values of K and T from the response curve.



(a)



(b)

Solution:

$$M_p = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} = 0.254$$

Take ln for both sides, we get:

$$ln\left(0.254 = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)}\right) \Rightarrow -1.37 = -\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)$$

Square both sides of the equation above, we get:

$$\left(-1.37 = -\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)\right)^2 \Rightarrow 1.8769 = \frac{(\zeta\pi)^2}{1-\zeta^2} \Rightarrow 1.8769\,(1-\zeta^2) = (\zeta\pi)^2$$

$$9.8596\,\zeta^2 + 1.8769\,\zeta^2 = 1.8769 \Rightarrow 11.7365\,\zeta^2 = 1.8769 \Rightarrow \zeta^2 = \frac{1.8769}{11.7365}$$

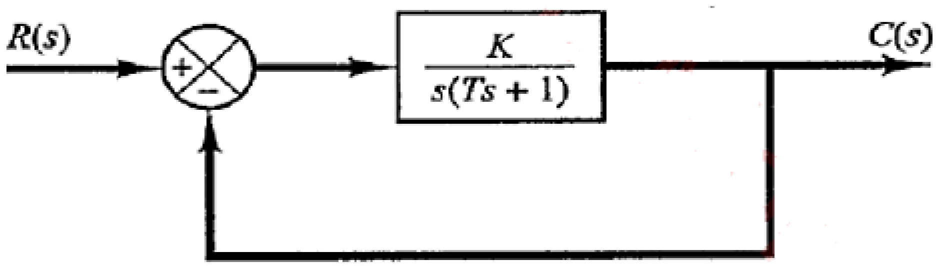
$$\zeta^2 = 0.1599 \Rightarrow \zeta = 0.3998 \Rightarrow \boldsymbol{\therefore \zeta \cong 0.4}$$

From the response curve, we have: $t_p = 3$

$$t_p = 3 = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \frac{\pi}{\omega_n\sqrt{1-0.4^2}} = \frac{3.14}{0.9165\,\omega_n}$$

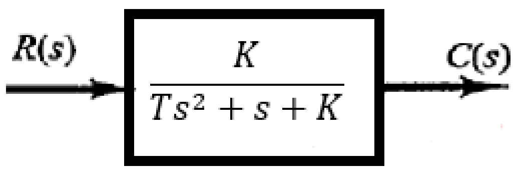
$$\because 3 = \frac{3.14}{0.9165\,\omega_n} \Rightarrow 2.7495\,\omega_n = 3.14 \Rightarrow \omega_n = \frac{3.14}{2.7495} \Rightarrow \boldsymbol{\omega_n = 1.14\,rad/sec}$$

For simplifying the block diagram in figure (a), use **Rule 3** of block diagram reduction rules for blocks $\frac{K}{s(Ts+1)}$ and unity feedback which are connected in negative feedback loop to combine them into one block.



(a)

After simplification, the block diagram of the system will be shown in figure below:



$$\frac{C(s)}{R(s)} = \frac{K}{Ts^2 + s + K}$$

Divide both sides of the equation above by (T), we get:

$$\frac{C(s)}{R(s)} = \frac{K/T}{s^2 + \frac{s}{T} + \frac{K}{T}}$$

Comparing the above equation with the standard second order transfer function shown below:

$$\frac{C(s)}{R(s)} = \frac{{\omega_n}^2}{s^2 + 2\zeta\omega_n s + {\omega_n}^2}$$

Therefore, the values of T and K are determined as:

$$2\zeta\omega_n = \frac{1}{T} \Rightarrow T = \frac{1}{2\zeta\omega_n} = \frac{1}{2 \times 0.4 \times 1.14} \Rightarrow \boldsymbol{\therefore T = 1.096}$$

$$And\ {\omega_n}^2 = \frac{K}{T} \Rightarrow K = {\omega_n}^2\, T$$

Therefore, the value of K is:

$\because K = \omega_n^2 T = 1.14^2 \times 1.096 \Longrightarrow \therefore \boldsymbol{K = 1.42}$

Homework1: A unity feedback system is characterized by an open loop transfer function:

$$\boldsymbol{G(s) = \frac{K}{s(s+10)}}$$

Determine the gain K so that the system will have a damping ratio of 0.5. For this value of K, determine settling time, peak percent overshoot, and time to peak overshoot for a unit step input.

Answer:

K=100

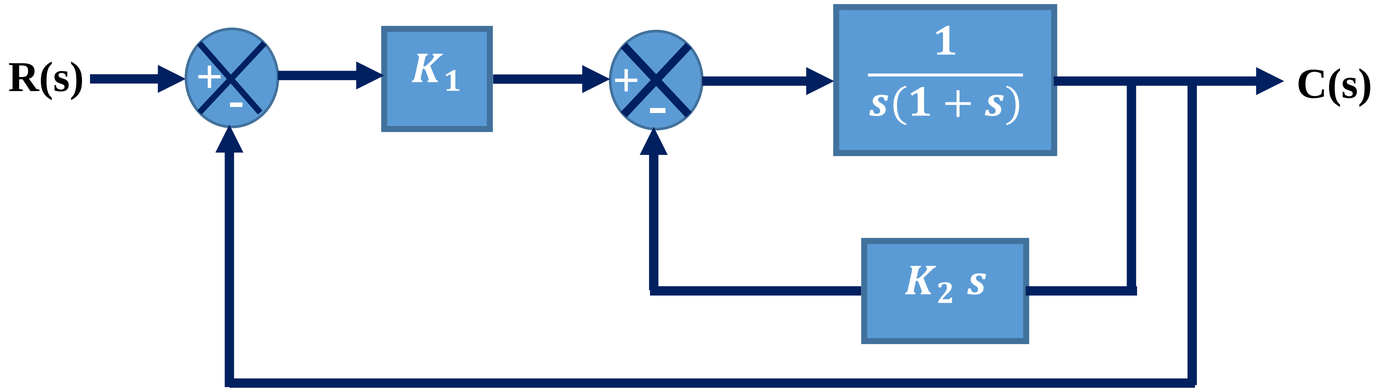
t_s=0.8 sec

Peak percent overshoot =16.3%

t_p=0.362 sec

Homework2: A feedback system employing output rate damping is shown in Figure below.

A. **Find the values of K_1 and K_2 so that the closed-loop system resembles a second-order system with damping ratio equal to 0.5 and frequency of damped oscillations 9.5 rad/sec.**
B. **With the above values of K_1 and K_2, find the percentage overshoot when input is step input.**
C. **What is the settling time for 2 percent tolerance?**



Answer:

K_1=120.34

K_2=9.97

M_P=16.31%

t_s=0.729 sec